JOURNAL OF APPROXIMATION THEORY 13, 158–166 (1975)

A Theorem on Nonempty Intersection of Convex Sets and its Application

T. ANDO

Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan

Communicated by P. L. Butzer

DEDICATED TO PROFESSOR G. G. LORENTZ
ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

1. INTRODUCTION

Several important properties in theory of Banach spaces can be formulated in terms of nonemptiness of a finite number of closed convex subsets. For instance, a Banach space, ordered with a closed cone K, is directed if and only if the intersection $K \cap (x + K)$ is nonempty for every vector x. As another example, consider a Banach space E and a closed subspace J. Then the image of a closed convex subset S under the quotient map from E to E/J is closed if and only if the intersection $S \cap (x + J)$ is nonempty for every x in the closure of $S + J$.

In discussing nonemptiness of intersection of convex subsets of a Banach space E, it is usually convenient to imbed E canonically into the second dual E^{**} and to use weak**, that is, $\sigma(E^{**}, E^{*})$, compactness of the unit ball of E^{**}. Given closed convex subsets S_i $(i = 1, 2)$ of E, sometimes the nonemptiness of $S_1^{\sim} \cap S_2^{\sim}$ is guaranteed quite easily, where $S_i^{\sim}$ denotes the weak** closure of S_i. The problem studied in this paper is to find a condition which transfers the nonemptiness of $S_1^{\sim} \cap S_2^{\sim}$ to that of $S_1 \cap S_2$.

In the next section we shall formulate a useful sufficient condition for nonemptiness of intersection (Theorem 1), which can be viewed as an abstract version of the so-called "$\frac{1}{2}^n$-technique." In the subsequent section this theorem is applied to reproduce various known results on images of closed convex sets under a quotient map. These results can be considered as general versions of the so-called "dominated extension theorems."

In the final section we shall treat a closed subspace J of a Banach space E whose weak** closure is the range of a projection in E^{**}. The problem in this section is to find a condition which assures that J itself is the range of a projection in E. On the basis of tensor product method, a variant of

Theorem 1 is effectively applied to yield general versions (Theorem 5) of the so-called "linear extension theorems."

2. Nonempty Intersection

Let E be a (real or complex) Banach space with closed unit ball U.

Before going into the subject, let us recall some computational rules for weak** closure. As stated in Introduction, the weak** closure $S^\sim$ is the closure of S in E^{**} with respect to the topology $\sigma(E^{**}, E^*)$ while S^- denotes the norm closure. Obviously $U^\sim$ is the closed unit ball of the second dual E^{**}, and $(S + U)^\sim = S^\sim + U^\sim$ and $S^\sim \cap E = S^-$ for every convex subset S. Given convex subsets S_1 and S_2 of E, if $S_1{}^\sim$ has nonempty intersection with $S_2{}^\sim$ then by the Hahn-Banach theorem $S_1 + \epsilon U$ has non-empty intersection with S_2 for every $\epsilon > 0$. If S_1 has nonempty intersection with the interior of S_2 then by the bipolar theorem $(S_1 \cap S_2)^\sim$ coincides with $S_1{}^\sim \cap S_2{}^\sim$.

Theorem 1. *Let S_i $(i = 1, 2)$ be closed convex subsets of E. If $S_1{}^\sim \cap S_2{}^\sim$ is nonempty and if there are constants $0 \leqslant \alpha, \beta$ and $0 \leqslant \gamma < 1$ such that*

$$S_1 \cap (S_2 + \lambda U) \subseteq \alpha\lambda U^\sim + (1 + \beta\lambda)\{S_1{}^\sim \cap (S_2{}^\sim + \gamma\lambda U^\sim)\} \qquad (\lambda > 0), \qquad (*)$$

then $S_1 \cap S_2$ is nonempty. If further $\beta = 0$ then

$$S_1 \cap (S_2 + \lambda U) \subseteq S_1 \cap S_2 + \alpha'\lambda(1 - \rho)^{-1} U \qquad (\lambda > 0, 1 > \rho > \gamma \text{ and } \alpha' > \alpha).$$

Proof. Take ρ with $\gamma < \rho < 1$ and α' with $\alpha < \alpha'$. Fix x_0 in $S_1 \cap (S_2 + \lambda U)$. Suppose that $x_0, \ldots, x_n$ can be chosen so as to satisfy the conditions:

$$x_j \in \delta_j\{S_1 \cap (S_2 + \lambda\rho^j U)\} \quad \text{and} \quad x_j - x_{j-1} \in \alpha'\lambda\delta_{j-1}\rho^{j-1} U \quad (j = 0, 1, \ldots, n)$$

where

$$x_{-1} = x_0, \delta_{-1} = \delta_0 = 1 \qquad \text{and} \qquad \delta_j = \prod_{k=0}^{j-1} (1 + \beta\lambda\rho^k).$$

Since by (*)

$$x_n \in \delta_n\{S_1 \cap (S_2 + \lambda\rho^n U)\} \subseteq \alpha\lambda\delta_n\rho^n U^\sim + \delta_{n+1}\{S_1{}^\sim \cap (S_2{}^\sim + \lambda\rho^{n+1} U^\sim)\},$$

it follows that $(x_n + \alpha\lambda\delta_n\rho^n U)^\sim$ has nonempty intersection with $\delta_{n+1}\{S_1 \cap (S_2 + \lambda\rho^{n+1} U)\}^\sim$. Since $x_n + \alpha'\lambda\delta_n\rho^n U$ is a convex neighborhood of $x_n + \alpha\lambda\delta_n\rho^n U$, there is a vector, say x_{n+1}, in the intersection of $x_n + \alpha'\lambda\delta_n\rho^n U$ and $\delta_{n+1}\{S_1 \cap (S_2 + \lambda\rho^{n+1} U)\}$. Then $x_{n+1} - x_n \in \alpha'\lambda\delta_n\rho^n U$.

This completes the inductive construction of a sequence $\{x_j\}$. Since $\sum_{n=0}^{\infty} \rho^n = (1-\rho)^{-1}$ and $1 \leqslant \delta_n \leqslant \exp(\beta\lambda(1-\rho)^{-1})$, $\{\delta_n^{-1}x_n\}$ is a Cauchy sequence such that $\delta_n^{-1}x_n \in S_1$ and $\| \delta_n^{-1}x_n - S_2 \| \leqslant \lambda\rho^n$. The limit x must belong to $S_1 \cap S_2$. Finally if $\beta = 0$ then $\delta_n = 1$, hence

$$\| x - x_0 \| \leqslant \sum_{n=1}^{\infty} \| x_n - x_{n-1} \| \leqslant \alpha'\lambda \sum_{n=0}^{\infty} \rho^n = \alpha'\lambda(1-\rho)^{-1}.$$

This completes the proof.

Let us give some elementary applications of Theorem 1 to theory of ordered Banach spaces. Now let E be a real Banach space, ordered by the closed convex cone E_+.

If $E_+{}^\sim$ generates E^{**}, that is, $E^{**} = E_+{}^\sim - E_+{}^\sim$ then E_+ generates E.

In fact, it is readily shown by contradiction that

$$U^\sim \subseteq E_+{}^\sim \cap \gamma U^\sim - E_+{}^\sim \cap \gamma U^\sim \qquad \text{for some} \quad \gamma > 0.$$

Take x in E. Then

$$(x + E_+) \cap (E_+ + \lambda U) \subseteq \gamma\lambda U^\sim + (x + E_+)^\sim \cap E_+{}^\sim \qquad (\lambda > 0).$$

Now Theorem 1 guarantees the nonemptiness of $(x + E_+) \cap E_+$, or equivalently $x \in E_+ - E_+$.

If E^{**} becomes a vector lattice (Riesz space) under the order induced by $E_+{}^\sim$ and if $(U^\sim - E_+{}^\sim) \cap (U^\sim + E_+{}^\sim)$ is bounded, then E has the Riesz interpolation property, that is, $x_i \leqslant y_j$ $(i, j = 1, 2)$ in E implies the existence of z in E with $x_i \leqslant z \leqslant y_i$ $(i = 1, 2)$.

In fact, with $S_1 = (x_1 + E_+) \cap (x_2 + E_+)$ and $S_2 = (y_1 - E_+) \cap (y_2 - E_+)$ it is readily shown that

$$S_1{}^\sim = \{z \in E^{**};\, z \geqslant x_1 \vee x_2\} \qquad \text{and} \qquad S_2{}^\sim = \{z \in E^{**};\, z \leqslant y_1 \wedge y_2\}.$$

Hence $S_1{}^\sim \cap S_2{}^\sim$ is nonempty and for some $\alpha > 0$,

$$S_1 \cap (S_2 + \lambda U) \subseteq \alpha\lambda U^\sim + S_1{}^\sim \cap S_2{}^\sim \qquad (\lambda > 0).$$

Now Theorem 1 guarantees nonemptiness of $S_1 \cap S_2$, or equivalently the existence of z in question.

In this way Theorem 1 can provide a unified way of proof for various duality properties developed, for instance, in [4] and [11].

3. Quotient

Again E is a Banach space with closed unit ball U. Let J be a closed subspace and let τ denote the quotient map from E to E/J. $J^\perp$ denotes the annihilator of J in E^*. The problem in this section is to find conditions which guarantee the closedness of $\tau(S)$ for a closed convex subset S of E. For this purpose, it is convenient to convert Theorem 1 to the following form.

Proposition 2. *Let S_i $(i = 1, 2)$ be closed convex subsets of E. If $J \subseteq S_1 \subseteq S_2^\sim + J^\sim$ and if there are constants $0 < \alpha$ and $0 \leqslant \gamma < 1$ such that*

$$S_1 \cap (S_2 + \lambda U) \subseteq \alpha\lambda(J^\sim \cap U^\sim) + S_2^\sim + \gamma\lambda U^\sim \qquad (\lambda > 0), \qquad (**)$$

then

$$S_1 \cap (S_2 + \lambda U) \subseteq S_2 + \alpha\lambda(1 - \rho)^{-1} (J \cap U) \qquad (\lambda > 0, 1 > \rho > \gamma).$$

In particular, $\tau(S_1)$ is contained in $\tau(S_2)$.

Proof. Take x in S_1 and let $S_0 = x + J$. Then $x \in S_2^\sim + J^\sim$ implies $S_0^\sim \cap S_2^\sim \neq \varnothing$. Now by (**)

$$S_0 \cap (S_2 + \lambda U) \subseteq S_1 \cap (S_2 + \lambda U) \subseteq \alpha\lambda(J^\sim \cap U^\sim) + S_2^\sim + \gamma\lambda U^\sim,$$

which implies

$$S_0 \cap (S_2 + \lambda U) \subseteq \alpha\lambda U^\sim + S_0^\sim \cap (S_2^\sim + \gamma\lambda U^\sim).$$

Then Theorem 1 yields

$$x \in S_0 \cap (S_2 + \lambda U) \subseteq S_2 + \alpha\lambda(1 - \rho)^{-1} (J \cap U).$$

This completes the proof.

As an immediate application of Proposition 1 to theory of ordered Banach spaces, let us take up the following proposition.

If a closed cone E_+ of a real Banach space E satisfies the condition that for some constant $\alpha \geqslant 1$ and for every positive linear functional f

$$\| f - [0, f] \cap J^\perp \| \leqslant \alpha \| f - J^\perp \|,$$

where $[0, f]$ denotes the set $\{g \in E^*;\, 0 \leqslant g \leqslant f\}$ then the image $\tau(E_+)$ is closed.

In fact, the bipolar theorem converts the above condition to the following inclusion relation:

$$(E_+ + J)^- \cap (E_+ + \lambda U) \subseteq \alpha\lambda(J^\sim \cap U^\sim) + E_+^\sim.$$

Apply Proposition 1 with $S_1 = (E_+ + J)^-$ and $S_2 = E_+$, to yield $(E_+ + J)^- \subseteq E_+ + J$.

A closed subspace is called a *summand* if it is the range of a (continuous linear) projection. In this respect a closed subspace J of E is called an *ideal* if its annihilator $J^\perp$ is a summand of E^*. Remark that J is an ideal if and only if there is a (continuous linear) operator T from E to the second dual E^{**} such that T vanishes on J and $x - Tx$ belongs to $J^\sim$ for every $x \in E$. In fact, if P is a projection from E^* to $J^\perp$ then the restriction of P^* to E possesses the required properties. Conversely if T is an operator in question and if Φ denotes the canonical imbedding of E^* to the third dual E^{***} then $T^* \circ \Phi$ is a projection to $J^\perp$.

Now let J be an ideal and let P be a projection from E^* to $J^\perp$. A closed convex subset S of E is said to be *splittable*, more precisely P-splittable, if it contains the origin and if $P^*x + y - P^*y$ belongs to $S^\sim$ for every x, y in S. Remark (cf. [5]) that S is splittable if and only if the polar S^0 coincides with the norm closed convex hull of $P(S^0) \cup Q(S^0)$ where $Q = 1 - P$.

A basic fact in [5] is that if S_i $(i = 1, 2)$ are splittable then

$$\begin{aligned}(S_1 + J)^- \cap (S_2 + J)^- \cap \{S_1 \cap (S_2 + \epsilon U)^- + \lambda U\} \\ \subseteq \alpha\lambda(J^\sim \cap U^\sim) + S_1^\sim \cap (S_2 + \alpha\epsilon U)^\sim \qquad (\lambda > 0, \epsilon > 0), \qquad (\#)\end{aligned}$$

where $\alpha = \| 1 - P \|$. Further if $(S_1 \cap S_2)^\sim$ coincides with $S_1^\sim \cap S_2^\sim$ then ϵ can be put equal to 0.

Let us reprove a main result of [5]:

PROPOSITION 3. *Let S_i $(i = 1, 2)$ be splittable. Then*

(1) *$\tau(S_i)$ is closed.*

(2) *If $(S_1 \cap S_2)^\sim = S_1^\sim \cap S_2^\sim$ then $\tau(S_1)^- \cap \tau(S_2)^- = \tau(S_1 \cap S_2)$.*

(3) *If $\| 1 - P \| \leqslant 1$ then $\tau(S_1) \cap \tau(S_2) \subseteq \tau(S_1 \cap (S_2 + \epsilon U))$ $(\epsilon > 0)$.*

Proof. (1) follows (2) with $S_1 = S_2$. (2) and (3) result from (#) by Proposition 2, applied to $(S_1 + J)^- \cap (S_2 + J)^-$ and $S_1 \cap (S_2 + \epsilon U)^-$.

Proposition 3 unifies so-called "dominated extension theorems" of various types developed in [2, 3, 7, 8] through suitable construction of P from an operator T as remarked earlier.

A closed subspace J is called an *M-ideal* if its annihilator $J^\perp$ is the range of a projection P such that

$$\| f \| = \| Pf \| + \| f - Pf \| \qquad (f \in E^*),$$

or equivalently

$$\| x \| = \max(\| P^*x \|, \| x - P^*x \|) \qquad (x \in E^{**}).$$

Alfsen and Effros [1] showed that a projection with this property is uniquely determined, and that an M-ideal J can be characterized by the following ball-intersection properties: if open balls O_i $(i = 1,..., n)$ have nonempty intersection and if each has nonempty intersection with J then $J \cap O_1 \cap O_2 \cap \cdots \cap O_n$ is nonempty.

It is clear that an ideal J is an M-ideal if and only if the unit ball is splittable. Now it follows as an immediate consequence of Proposition 3:

If J is an M-ideal then $\tau(U)$ is closed.

Application of Proposition 3 to various problems in ordered Banach spaces can be found in [5].

Finally let us give a simple proof to a result in [5] on the basis of Proposition 2.

If J is an M-ideal and if N is a closed subspace such that for some constant $0 \leqslant \gamma < 1$

$$\| Pg - J^{\perp} \cap N^{\perp} \| \leqslant \gamma \| g - Pg \| \qquad (g \in N^{\perp}),$$

then for $x \in N$ with $\| \tau(x) \| \leqslant 1$ there is $y \in N$ such that $\| y \| \leqslant 1$ and $\tau(x) = \tau(y)$.

In fact, the bipolar theorem converts the condition to the relation

$$N \cap (J + U)^{-} \cap (1 + \lambda) U \subseteq 2\lambda \{U^{\sim} \cap (N \cap J)^{\sim}\} + (1 + \gamma\lambda)\, U^{\sim}.$$

Now apply Proposition 2 with $S_1 = N \cap (J + U)^{-}$ and $S_2 = U$ and with $N \cap J$ instead of J.

4. Linear Lifting

An ideal J of a Banach space E becomes a summand if and only if the quotient map τ from E to E/J has a (continuous) right inverse. Indeed, a right inverse ζ gives rise to the projection $\zeta \circ \tau$ whose kernel coincides with J. The problem in this section is to find conditions which assure the existence of a right inverse.

When H is a subspace of E/J, a (continuous linear) operator ψ from H to E is called a *linear lifting* if $\tau \circ \psi$ is the identity on H. Let $\mathfrak{B}(H, E)$ denote the space of (continuous linear) operators from H to E, equipped with operator norm. If H is finite dimensional, the second dual of $\mathfrak{B}(H, E)$ is identified with $\mathfrak{B}(H, E^{**})$.

A Banach space is called a *Lindenstrauss space* if its second dual is isometric to the space of continuous functions on a compact Hausdorff space. Lindenstrauss [9] gave an intrinsic characterization of such a space by the following intersection property: a finite number of closed balls has nonempty intersection whenever every pair among them has non-empty intersection.

PROPOSITION 4. *Let J be an M-ideal and let H be a finite dimensional subspace of E/J. Suppose that ζ_0 is a linear lifting from a subspace G of H with $\| \zeta_0 \| \leqslant 1$. Then for each $\epsilon > 0$ there is a linear lifting ζ from H such that $\| \zeta \| \leqslant 1$ and $\| \zeta_0 - \zeta |_G \| < \epsilon$ if one of the following conditions is satisfied*:

(1) *there is a projection π from H to G with $\| \pi \| \leqslant 1$.*

(2) *J is a Lindenstrauss space.*

Proof. Since J is an M-ideal its annihilator $J^\perp$ is the range of a (uniquely determined) projection P such that

$$\| x \| = \max(\| P^*x \|, \| x - P^*x \|) \qquad (x \in E^{**}).$$

Consider two subspaces in $\mathfrak{B}(H, E)$

$$\mathfrak{M} = \{\psi; \operatorname{ran}(\psi) \subseteq J \text{ and } \ker(\psi) \supseteq G\} \qquad \text{and} \qquad \mathfrak{N} = \{\psi; \operatorname{ran}(\psi) \subseteq J\},$$

and let $\mathfrak{B}$ denote the closed unit ball of $\mathfrak{B}(H, E)$. As shown in [5], under the identification of $\mathfrak{B}(H, E^{**})$ with the second dual of $\mathfrak{B}(H, E)$, the weak** closures of $\mathfrak{M}$ and $\mathfrak{N}$ coincide, respectively, with

$$\{\psi; \operatorname{ran}(\psi) \subseteq J^\sim \text{ and } \ker(\psi) \supseteq G\} \qquad \text{and} \qquad \{\psi; \operatorname{ran}(\psi) \subseteq J^\sim\}.$$

Since $\psi \mapsto P^* \circ \psi$ defines a projection in $\mathfrak{B}(H, E^{**})$ with kernel $\mathfrak{N}^\sim$ such that

$$\| \psi \| = \max(\| P^* \circ \psi \|, \| \psi - P^* \circ \psi \|) \qquad (\psi \in \mathfrak{B}(H, E^{**})),$$

$\mathfrak{N}$ is an M-ideal. Take a linear lifting ζ_1 from H to E with $\zeta_1 |_G = \zeta_0$. Suppose that ζ_1 belongs to $(\mathfrak{B} + \mathfrak{M})^\sim$. Then there is an operator ζ_2 in $(\zeta_1 + \mathfrak{M}) \cap (1 + \epsilon/2)\mathfrak{B}$, which is contained in $(\mathfrak{B} + \mathfrak{N})^- \cap (1 + \epsilon/2)\mathfrak{B}$. Since $\mathfrak{N}$ is an M-ideal, in view of Propositions 2 and 3 $(\mathfrak{B} + \mathfrak{N})^- \cap (1 + \epsilon/2)\mathfrak{B}$ is contained in $\mathfrak{B} + \epsilon(\mathfrak{N} \cap \mathfrak{B})$. Thus there is ζ such that $\| \zeta \| \leqslant 1$, $\| \zeta - \zeta_2 \| \leqslant \epsilon$ and $\zeta - \zeta_2 \in \mathfrak{N}$. Then ζ meets the requirement of the assertion. Now it remains to prove that ζ_1 is contained in $(\mathfrak{B} + \mathfrak{M})^\sim$.

(1) Suppose that there is a projection π from H to G with $\| \pi \| \leqslant 1$. Then it follows with $Q = 1 - P$ that

$$\zeta_1 = (P^* \circ \zeta_1 + Q^* \circ \zeta_1 \cdot \pi) + Q^* \circ \zeta_1 \circ (1 - \pi) \in \mathfrak{B}^\sim + \mathfrak{M}^\sim = (\mathfrak{B} + \mathfrak{M})^\sim.$$

(2) Suppose finally that J is a Lindenstrauss space. Since $J^\sim$ is isometric to the second dual of the Lindenstrauss space J it follows readily that there is a linear operator ψ from H to $J^\sim$ such that $\psi |_G = \zeta_0 - P^* \circ \zeta_0$ and $\| \psi \| = \| \zeta_0 - P^* \circ \zeta_0 \|$. Since

$$\| P^* \circ \zeta_1 + \psi \| \leqslant \max(\| P^* \circ \zeta_1 \|, \| \psi \|) \leqslant 1,$$

it follows that

$$\zeta_1 = (P^* \circ \zeta_1 + \psi) + (\zeta_1 - P^* \circ \zeta_1 - \psi) \in \mathfrak{B}^\sim + \mathfrak{M}^\sim = (\mathfrak{B} + \mathfrak{M})^\sim.$$

This completes the proof.

The next theorem is close, in its spirit, to the results of Asimov [6], Michael and Pełczyński [10] and Rao [13].

THEOREM 5. *Let J be a closed subspace of a Banach space E such that the quotient space E/J is separable and that the annihilator $J^\perp$ is the range of a projection P satisfying, for some constant $\epsilon > 0$,*

$$\|f\| \geqslant \epsilon \| Pf \| + \|f - P^*f\| \qquad (f \in E^*). \tag{\$}$$

Then J is a summand if one of the following conditions is satisfied:

(1) *the quotient space E/J has the bounded approximation property, that is, the identity operator in E/J is in the strong closure of a bounded set of finite rank operators*

(2) *J is a Lindenstrauss space.*

Proof. On the basis of ($) and the Krein–Smulian theorem it is easy to prove that the convex hull of $\epsilon^{-1}(U^0 \cap J^\perp) \cup (1 - P)(U^0)$ is weak* closed where U^0 is the unit ball of E^*. Therefore there is a norm on E which is equivalent to the original norm and coincides with the original norm on J and for which the relation, corresponding to ($), is valid with $\epsilon = 1$. Thus it is assumed without loss of generality that J is an M-ideal.

(1) Suppose that E/J has the bounded approximation property. Then in view of separability there is a sequence $\{\pi_n\}$ of finite rank operators, converging strongly to the identity. Consider the space $c(E)$ of E-valued converging sequences. The spaces $c(J)$ and $c(E/J)$ are defined correspondingly. It is readily shown that $c(J)$ is considered as an M-ideal of $c(E)$ and that $c(E)/c(J)$ is canonically identified with $c(E/J)$. Consider the subspace $\mathfrak{F}$ of $c(E/J)$, consisting of vectors $\{\hat{x}_n\}$ such that $\hat{x}_{n+1} - \hat{x}_n$ belongs to the range of $\pi_{n+1} - \pi_n$ for every $n \geqslant 1$. The correspondence $\Pi_m\{\hat{x}_n\} = \{\hat{y}_n\}$, defined by $\hat{y}_n = \hat{x}_{\min[n,m]}$, is a linear projection in $\mathfrak{F}$ with $\| \Pi_m \| \leqslant 1$. Obviously Π_m converges strongly to the identity in $\mathfrak{F}$. Then successive application of Proposition 4 (1) shows, as in (2) below, that there is a linear lifting Φ from $\mathfrak{F}$ to $c(E)$. Since for each $\hat{x}$ in E/J the sequence $\{\pi_n(\hat{x})\}$ belongs to $\mathfrak{F}$ the linear operator $\zeta(\hat{x})$, defined as the valued of $\Phi(\{\pi_n(x)\})$ at infinity, gives rise to a linear lifting from E/J to E.

(2) Suppose that J is a Lindenstrauss space. Successive approximation of Proposition 4 (2) shows that there is a sequence $\{G_n\}$ of finite-dimensional

subspaces of E/J such that $G_n \subseteq G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n$ is dense in E/J, and there is a sequence $\{\zeta_n\}$ such that ζ_n is a linear lifting from G_n with $\| \zeta_n \| \leq 1$ and $\| \zeta_n - \zeta_{n+1} |_{G_n} \| \leq \frac{1}{2}^n$. Then the limit ζ of ζ_n can be unambiguously defined on $\bigcup_{n=1}^{\infty} G_n$. Since $\| \zeta \| \leq 1$, the continuous extension of ζ gives a desired linear lifting from E/J. This completes the proof.

The above proof for (1) is inspired by a method in Pełczyński and Wojtaszczyk [12].

References

1. E. M. Alfsen and E. G. Effros, Structure in real Banach spaces I–II, *Ann. Math.* **96** (1972), 98–173.
2. E. M. Alfsen and B. Hirsberg, On dominated extensions in linear subspaces of $C(X)$, *Pacific J. Math.* **36** (1971), 567–584.
3. T. B. Andersen, On dominated extensions of continuous affine functions on split faces, *Math. Scand.* **29** (1971), 298–306.
4. T. Ando, On fundamental properties of a Banach space with a cone, *Pacific J. Math.* **12** (1962), 1163–1169.
5. T. Ando, Closed range theorems for convex sets and linear liftings, *Pacific J. Math.* **44** (1973), 393–410.
6. L. Asimov, Affine selections on simplicially split convex sets, *J. London Math. Soc.* **6** (1973), 533–538.
7. E. Briem, Restrictions of subspaces of $C(X)$, *Invent. Math.* **10** (1970), 288–297.
8. E. Briem and M. Rao, Normpreserving extensions in subspaces of $C(X)$, *Pacific J. Math.* **42** (1972), 581–587.
9. J. Lindenstrauss, Extension of compact operators, *Mem. Amer. Math. Soc.* **48** (1964), 1–112.
10. E. Michael and A. Pełczyński, A linear extension theorem, *Illinois J. Math.* **11** (1967), 563–579.
11. K.-F. Ny, The duality of partially ordered Banach spaces, *Proc. London Math. Soc.* **19** (1969), 267–288.
12. A. Pełczyński and P. Wojtaszczyk, Banach spaces with finite dimensional expansions of identity and universal bases of finite dimensional subspaces, *Studia Math.* **40** (1971), 91–108.
13. M. Rao, On simultaneous extensions, *Invent. Math.* **13** (1971), 284–294.